Worcester County Mathematics League

Varsity Meet 3: January 19, 2022

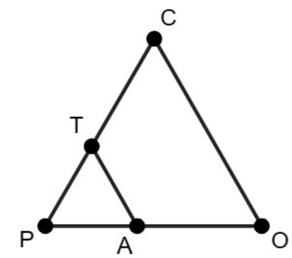
COACHES' COPY

ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 3 - January 19, 2022
Round 1 - Similarity and Pythagorean Theorem

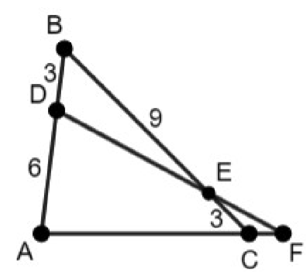
All answers must be in simplest exact form in the answer section.

1. $OC = 7$ in equilateral triangle ΔCOP shown at right, with $PT = PA = 3$. Find the perimeter of quadrilateral $COAT$.



2. Given a rhombus with area equal to 20 and sides of length $\sqrt{29}$, find the length of the longer diagonal of the rhombus.

3. ΔABC shown at right is cut by line $\overleftrightarrow{DE}$ such that $AD = 6, DB = 3, BE = 9$ and $EC = 3$. If AC is extended to meet $\overleftrightarrow{DE}$ at point F, find the ratio $CF : AF$ and express it as $m : n$, where m and n are integers whose greatest common factor is 1.



ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. __________ : __________

Burncoat, Shrewsbury, Doherty

All answers must be in simplest exact form in the answer section.

1. A mixture of olive oil and vinegar contains two parts olive oil to three parts vinegar. In what ratio should this mixture be combined with pure olive oil to produce a new mixture with two parts olive oil and one part vinegar? Write your answer in the simplest integer ratio $m : n$ with m corresponding to the quantity of the original mixture and n corresponding to the quantity of the added olive oil.

2. The expression $7w + 8x - 3(23w + 34y - 9z - 9(3x + 3y - 5z + 5(3w + 4x + 2y - 7z)))$ can be simplified to the form $Aw + Bx + Cy + Dz$. Find the value of $A + B + C + D$.

3. Positive integers x, y and k are chosen such that $x < y$ and $\frac{10x}{x+y} + \frac{20y}{x+y} = k$. How many possible values are there for k?

ANSWERS

(1 pt) 1. __________ : __________

(2 pts) 2. ____________________

(3 pts) 3. ____________________ Algonquin, Mass Academy, Shrewsbury

Worcester County Mathematics League
Varsity Meet 3 - January 19, 2022
Round 3 - Functions

All answers must be in simplest exact form in the answer section.

1. Let $f(x) = -2x + 7$. Find $f^{-1}(-4)$. Express your answer as a simplified improper faction $\dfrac{m}{n}$, where m and n are integers.

2. Let $f(x) = \sqrt{3 - 2x}$ and $g(x) = f(f(x))$. Find the domain of $g(x)$. Express your answer in interval notation (for example $x \in (-2, 3]$ indicates that $-2 < x \leq 3$.)

3. Let $f(x)$ be a periodic function such that $f(x - 2) = f(x)$ and $f(x) = x^2$ for $x \in [-1, 1]$. Let $g(x) = 2f\left(\frac{x}{3} + \frac{1}{2}\right) + 1$. Find $\frac{g(2021)}{g(2022)}$.

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

AMSA, Bromfield, QSC

Worcester County Mathematics League
Varsity Meet 3 - January 19, 2022
Round 4 - Combinatorics

All answers must be in simplest exact form in the answer section.

1. In how many different ways can a student answer a 5 question multiple choice test if there are 4 choices for each question?

2. The natural numbers from 1 through 18 are grouped into nine pairs of numbers such that the sum of each pair is a perfect square. Let k, l, m, and n be the numbers of pairs that sum to 4, 9, 16, and 25, in that order. Find the ordered 4-tuple (k, l, m, n).

3. How many positive even integers less than 500 can be formed using only the digits 1,2,3,4,5,6 if a digit must be repeated and no digit appears more than twice?

ANSWERS

(1 pt) 1. ________________

(2 pts) 2. (____ , ____, ____, ____)

(3 pts) 3. ________________

Auburn, North, Tantasqua

All answers must be in simplest exact form in the answer section.

1. Find the length of the longest side of ΔANT, with the vertices $A(-1,2)$, $N(3,5)$, $T(-1,8)$.

2. Find the distance from the point $P(3,-11)$ to the line with the equation $5x-12y=17$.

3. Recall that a parabola is the set of points equidistant from its focus (a point) and its directrix (a line). Find the equation for the parabola with focus $(3,4)$ and directrix $y=-x$. If the equation is of the form $x^2+Ay^2+Bx+Cy+Dxy=E$, find $A+B+C+D+E$.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. (__________ , __________)

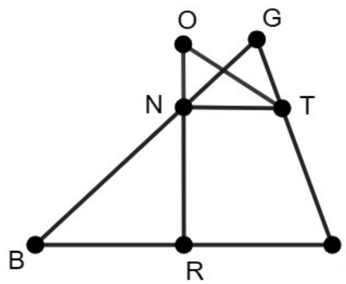
Bartlett, Quaboag, QSC

Worcester County Mathematics League

Varsity Meet 3 - January 19, 2022

Team Round

All answers must be in simplest exact form in the answer section.



1. Given ΔBIG shown at right, with $\overline{BI} \parallel \overline{NT}, BN = 9, NG = 6, BI = 10, OT = 5, ON = 3$, and $BR = 3\sqrt{5}$. Find the area of quadrilateral $BITN$.

2. Let $f(x) = x^2$, $g(x) = 1 - x$, $h(x) = 1/x$, and $k(x) = \sin(x)$. Express $\mu(x) = 1 + \tan^2(x)$ as a compositon involving one or more of the functions f, g, h, and k.

3. If the rational expression $\frac{2x-9}{x^2-x-6}$ is decomposed into the sum $\frac{A}{x-3} + \frac{B}{x+2}$, find the ordered pair (A, B)

4. Find the point(s) P on the x-axis such that the line through P and $(1, 1)$ is perpendicular to the line through P and $(3, -8)$.

5. In how many different ways can 35¢ be made up from coins of denominations 1¢, 5¢, 10¢, and 25¢?

6. A stream runs due east to west. Alice is 7 miles to the north of the stream and Bill is 11 miles east and 3 miles south of Alice. What is the shortest distance Alice needs to travel in order to get to the stream and then to Bill?

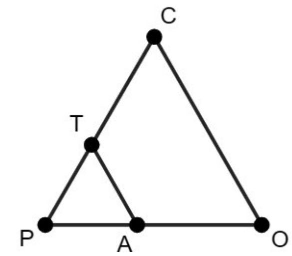
7. Two men working together can build an electric car in $3\frac{3}{7}$ weeks. If one of them works alone it would take him 6 weeks. How long would it take the other man working alone?

8. Consider the graph of an ellipse centered at $(0, a)$ and passing through the origin and $(1, a)$ for some value a. The range of values of a for which the ellipse has more than one point of intersection with the parabola $y = x^2$ can be expressed in interval notation as (c, d). Find the ordered pair (c, d).

9. How many distinct ways can 7 charms be arranged on a bracelet that has a clasp on it?

Quaboag, St. Peter-Marian, Shepherd Hill, Doherty, Tahanto, AMSA, Shrewsbury, Mass Academy, Shrewsbury

Round 1 - Sim. and the Pyth. Theorem

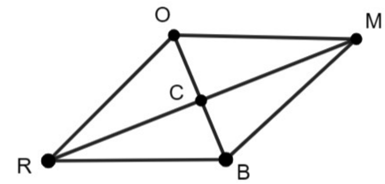


1. $OC = 7$ in equilateral triangle ΔCOP shown at right, with $PT = PA = 3$. Find the perimeter of quadrilateral $COAT$.

Solution: Since ΔCOP is equilateral, $OC = CP = PO = 7$ and $OA = PO - PA = 7 - 3 = 4$. Likewise, $TC = PC - PT = 7 - 3 = 4$. Now, because $\overline{PA} \cong \overline{PT}$ and $\angle P$ is shared, $\Delta TAP \sim \Delta COP$ by SAS similarity. Thus, ΔTAP is also equilateral and $AT = 3$. Finally, the perimeter of $COAT$ equals $CO + OA + AT + TC = 7 + 4 + 3 + 4 = \boxed{18}$.

2. Given a rhombus with area equal to 20 and sides of length $\sqrt{29}$, find the length of the longer diagonal of the rhombus.

Solution:



Rhombus $ROMB$ with sides of length $\sqrt{29}$ is shown above. Recall that the diagonals of a parallelogram bisect each other. Because a rhombus is a parallelogram, diagonals $\overline{RM}$ and $\overline{OB}$ of $ROMB$ bisect each other, with the point of intersection labeled C in the diagram. Next, note that the diagonals divide $ROMB$ into four congruent triangles (ΔROC, ΔMOC, ΔRBC and ΔMBC) because all three sides of the triangles are congruent (SSS postulate). These triangles are right triangles because any two adjacent angles centered at C form a line and are congruent, so their angle measure θ is found by $\theta + \theta = 180°$ and $\theta = 90°$.

Let $d_1 = RM$ and $d_2 = OB$ be the lengths of the two diagonals. The area of $ROMB$ is equal to four times the area of any one of the four right triangles. The lengths of the legs of one triangle are $\frac{d_1}{2}$ and $\frac{d_2}{2}$. The area of one right triangle is $\frac{1}{2}\frac{d_1}{2}\frac{d_2}{2} = \frac{d_1 d_2}{8}$, and the area of $ROMB$ is $4 \cdot \frac{1}{8} \cdot d_1 d_2 = \frac{d_1 d_2}{2} = 20$. Applying the Pythagorean theorem, $\left(\frac{d_1}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2 = \left(\sqrt{29}\right)^2 = 29$, that is, or $\frac{(d_1)^2 + (d_2)^2}{4} = 29$, or $(d_1)^2 + (d_2)^2 = 4 \cdot 29 = 116$. Now the diagonal lengths d_1 and d_2 are found by solving the system of equations:

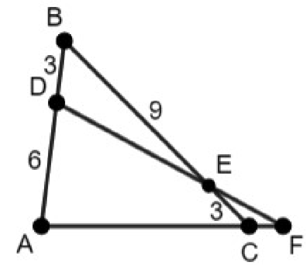
$$\begin{aligned} d_1 d_2 &= 40 \\ (d_1)^2 + (d_2)^2 &= 116 \end{aligned}$$

Add twice the top equation to the second equation to create perfect squares on each side of the equation:

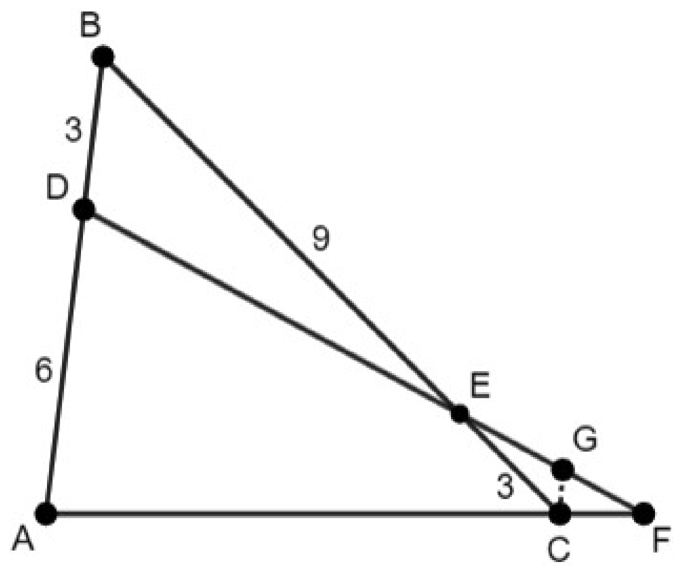
$$\begin{aligned} 2d_1 d_2 &= 80 \\ (d_1)^2 + 2d_1 d_2 + (d_2)^2 &= 116 + 80 \\ (d_1 + d_2)^2 &= 196 = 14^2 \\ d_1 + d_2 &= 14 \end{aligned}$$

The sum of the diagonal lengths is 14 and the product is 40. A search through all factors of 40 will find that $4 \cdot 10 = 40$ and $4 + 10 = 14$, so the diagonals $(d_1, d_2) = (4, 10)$ and the longer diagonal length $= \boxed{10}$.

3. ΔABC shown at right is cut by line $\overleftrightarrow{DE}$ such that $AD = 6, DB = 3, BE = 9$ and $EC = 3$. If AC is extended to meet $\overleftrightarrow{DE}$ at point F, find the ratio $CF : AF$ and express it as $m : n$, where m and n are integers whose greatest common factor is 1.



Solution:



Construct auxiliary line segment $\overline{CG}$ parallel to $\overline{AB}$ with G lying on $\overline{DF}$ as shown in the figure above. This line segment creates two pairs of similar triangles: $\Delta DBE \sim \Delta GCE$ and $\Delta ADF \sim \Delta CGF$. The desired ratio can be found using two proportions. From the first similarity:

$$BD : BE = CG : CE$$
$$3 : 9 = CG : 3$$

and $CG = 1$. From the second similarity,

$$CG : AD = CF : AF$$

Inserting the known values for CG and AD results in $CF : AF = \boxed{1 : 6}$.

Round 2 - Algebra I

1. A salad dressing contains two parts olive oil to three parts vinegar. In what ratio should this mixture be combined with pure olive oil to produce a new mixture with two parts olive oil and one part vinegar? Write your answer in the simplest integer ratio $m : n$ where m corresponds to the quantity of the original mixture and n corresponds to the quantity of the added olive oil.

Solution: The original mixture is in the ratio $2 : 3$, or $2x : 3x$, where $3x$ is the unknown amount of vinegar. Let y be the amount of added olive oil that is added to achieve the desired proportion:

$$(2x + y) : 3x = 2 : 1$$

Note that the quantity of olive oil in the new mixture is equal to the sum of the original quantity $2x$ and the added quantity y. Solving this proportion yields:

$$2x + y = 2 \cdot 3x = 6x$$

or $y = 6x - 2x = 4x$. Now the original quantity of the mixture is $2x + 3x = 5x$, so the ratio of the original mixture to the added olive oil is $5x : 4x$, or $\boxed{5 : 4}$.

2. The expression $7w + 8x - 3(23w + 34y - 9z - 9(3x + 3y - 5z + 5(3w + 4x + 2y - 7z)))$ can be simplified to the form $Aw + Bx + Cy + Dz$. Find the value of $A + B + C + D$.

Solution: The simplest solution is to note that the given expression is a function of the four variables w, x, y and z, specifically $f(w, x, y, z) = Aw + Bx + Cy + Dz$. Then $f(1, 1, 1, 1) = A + B + C + D$ and the problem is solved by substituting 1 for each of the four variables:

$$\begin{aligned}
f(1,1,1,1) &= 7(1) + 8(1) - 3(23(1) + 34(1) - 9(1) - 9(3(1) + 3(1) - 5(1) \\
&\qquad + 5(3(1) + 4(1) + 2(1) - 7(1))) \\
&= 7 + 8 - 3(23 + 34 - 9 - 9(3 + 3 - 5 + 5(3 + 4 + 2 - 7))) \\
&= 7 + 8 - 3(23 + 25 - 9(3 + 3 - 5 + 5(2))) \\
&= 7 + 8 - 3(48 - 9(6 - 5 + 10)) \\
&= 7 + 8 - 3(48 - 9(11)) \\
&= 15 - 3(48 - 99) \\
&= 15 - 3(-51) = 15 + 153 = \boxed{168}
\end{aligned}$$

3. Positive integers x, y and k are chosen such that $x < y$ and $\frac{10x}{x+y} + \frac{20y}{x+y} = k$. How many possible values are there for k?

Solution: First note that $0 < \dfrac{x}{y} < 1$ because x and y are positive and $x < y$. The problem can then be readily solved once the original equation is manipulated so that k is a function of the fraction $\dfrac{x}{y}$.

$$\begin{aligned} k &= \frac{10x}{x+y} + \frac{20y}{x+y} \\ &= \frac{10x+10y}{x+y} + \frac{10y}{x+y} \\ &= 10 + \frac{10y}{x+y} \\ &= 10 + \frac{10}{1+\frac{x}{y}} \end{aligned}$$

Now the inequality bounding $\dfrac{x}{y}$ between 0 and 1 can be manipulated to find an equivalent bound for k:

$$\begin{aligned} 0 < &\frac{x}{y} < 1 \\ 1 < 1 + &\frac{x}{y} < 2 \\ 1 > &\frac{1}{1+\frac{x}{y}} > \frac{1}{2} \\ 10 > &\frac{10}{1+\frac{x}{y}} > 10 \cdot \frac{1}{2} = 5 \\ 10 + 10 > 10 + &\frac{10}{1+\frac{x}{y}} > 10 + 5 \\ 20 > &k > 15 \end{aligned}$$

where the expression derived above was substituted for k in the last step. The possible values for integer k are 16, 17, 18, and 19. There are $\boxed{4}$ possible values of k.

Round 3 - Functions

1. Let $f(x) = -2x + 7$. Find $f^{-1}(-4)$. Express your answer as a simplified improper faction $\frac{m}{n}$, where m and n are integers.

> **Solution:** Note that $f\left(f^{-1}(x)\right) = x$ by the definition of an inverse function. Substituting $f^{-1}(x)$ for x in the original equation for $f(x)$ yields:
>
> $$f\left(f^{-1}(x)\right) = x = -2f^{-1}(x) + 7$$
>
> Adding $2f^{-1}(x) - x$ to both sides of this equation results in:
>
> $$2f^{-1}(x) = -x + 7$$
> $$f^{-1}(x) = \frac{-x+7}{2}$$
>
> And substituting $x = -4$ results in $f^{-1}(-4) = \frac{-(-4)+7}{2} = \boxed{\frac{11}{2}}$.

2. Let $f(x) = \sqrt{3-2x}$ and $g(x) = f(f(x))$. Find the domain of $g(x)$. Express your answer in interval notation (for example $x \in (-2, 3]$ indicates that $-2 < x \leq 3$.)

> **Solution:** For x to be in the domain of $g(x)$, it must be in the domain of $f(x)$. The domain of $f(x)$ is restricted by the square root function, so $3 - 2x \geq 0$. Then $2x \leq 3$, or $x \leq \frac{3}{2}$.
>
> Next, $f(f(x)) = \sqrt{3 - 2\sqrt{3-2x}}$ and the argument of the outer square root function must be non-negative, so:
>
> $$3 - 2\sqrt{3-2x} \geq 0$$
> $$3 \geq 2\sqrt{3-2x}$$
>
> Squaring both sides of the inequality (both sides are non-negative, so there are no extraneous solutions):
>
> $$9 \geq 4(3-2x) = 12 - 8x$$
> $$8x \geq 12 - 9 = 3$$
> $$x \geq \frac{3}{8}$$
>
> Combining the two inequalities, $\frac{3}{8} \leq x \leq \frac{3}{2}$, and the domain is $x \in \boxed{\left[\frac{3}{8}, \frac{3}{2}\right]}$.

3. Let $f(x)$ be a periodic function such that $f(x-2) = f(x)$ and $f(x) = x^2$ for $x \in [-1, 1]$. Let $g(x) = 2f\left(\frac{x}{3} + \frac{1}{2}\right) + 1$. Find $\frac{g(2021)}{g(2022)}$.

Solution: Calculating the denominator $g(2022)$:

$$\begin{aligned} g(2022) &= 2f\left(\frac{2022}{3} + \frac{1}{2}\right) + 1 \\ &= 2f\left(674 + \frac{1}{2}\right) + 1 \\ &= 2f\left(\frac{1}{2}\right) + 1 \end{aligned}$$

Note that due to the periodicity of $f(x)$, adding or subtracting any even number from its argument does not change the result. Continuing by applying the rule for $f(x)$:

$$\begin{aligned} g(2022) &= 2\left(\frac{1}{4}\right) + 1 \\ &= \frac{1}{2} + 1 \\ &= \frac{3}{2} \end{aligned}$$

The numerator $g(2021)$ is calculated likewise:

$$\begin{aligned} g(2021) &= 2f\left(\frac{2021}{3} + \frac{1}{2}\right) + 1 \\ &= 2f\left(673\frac{2}{3} + \frac{1}{2}\right) + 1 \\ &= 2f\left(673\frac{2}{3} - 674 + \frac{1}{2}\right) + 1 \\ &= 2f\left(-\frac{1}{3} + \frac{1}{2}\right) + 1 \\ &= 2f\left(\frac{1}{6}\right) + 1 \\ &= 2\left(\frac{1}{36}\right) + 1 \\ &= \frac{1}{18} + 1 \\ &= \frac{19}{18} \end{aligned}$$

Finally, $\dfrac{g(2021)}{g(2022} = \dfrac{19}{18} \div \dfrac{3}{2} = \dfrac{19}{18} \cdot \dfrac{2}{3} = \boxed{\dfrac{19}{27}}$.

Round 4 - Combinations

1. In how many different ways can a student answer a 5 question multiple choice test if there are 4 choices for each question?

Solution: The choices of answers for each question are independent. The number of different ways to answer the test is the product of four choices for the first question, 4 choices for the question 2, 4 choices for question 3, 4 choices for question 4 and 4 choices for question 5, or $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$, or $4^5 = \boxed{1024}$.

2. The natural numbers from 1 through 18 are grouped into nine pairs of numbers such that the sum of each pair is a perfect square. Let k, l, m, and n be the numbers of pairs that sum to 4, 9, 16, and 25, in that order. Find the ordered 4-tuple (k, l, m, n).

Solution: Note that $18 + 7 = 25$, and that 7 is the only natural number less than 18 that will sum to a square with 18. Likewise, $17 + 8 = 16 + 9 = 25$. Of the remaining twelve numbers, $1 + 15 = 2 + 14 = 3 + 13 = 4 + 12 = 5 + 11 = 6 + 10 = 16$. Thus, $(k, l, m, n) = \boxed{(0, 0, 6, 3)}$.

Note that this pairing of the numbers from 1 through 18 is the only pairing with the desired property. If, for instance, the pairing is changed so that 15 is paired with 10, then their partners 1 and 6 must be paired and $1 + 6 = 7$ is not a perfect square. The same is true for $(11, 14)$ and $(12, 13)$. Changing the pairing so that there are two or three additional pairs summing to 25 are chosen also doesn't work. The number 2 must be paired with 14; the only potential double pairing of 25 sums, $(10, 15), (12, 13)$, leaves $(1, 3, 4, 6)$ to be paired off; and there will be at least one pair $(1 + 4$ or $4 + 6)$ whose sum is not a perfect square.

3. How many positive even integers less than 500 can be formed using only the digits 1,2,3,4,5,6 if a digit must be repeated and no digit appears more than twice?

Solution: There are only four possible patterns for the digits of the numbers: aa, aba, aab, baa, with a and b being distinct numbers from 1 to 6. The problem can be solved case by case. Case I (aa): there are 3 two digit numbers that fit the criteria: 22, 44, and 66 (they must be even). Case II (aba): there are two choices for a (2,4) because it must be even and less than 5. Once a is chosen, there are five choices for b left, so there are $2 \cdot 5 = 10$ numbers with this pattern that fit the criteria. Case III (baa): If b is odd (either 1 or 3), then there are three choices for a (2,4,6). If b is even (either 2 or 4), there are only two remaining choices for a (6 and either 2 or 4). Thus, there are $2 \cdot 3 + 2 \cdot 2 = 10$ numbers with this pattern that fit the criteria. Case IV (aab): The same argument given in Case III applies to this case, switching a for b. Then there are 10 numbers in Case IV, as well.

Summing up the totals for the four cases, there are $3 + 10 + 10 + 10 = \boxed{33}$ numbers that fit the criteria.

Round 5 - Analytic Geometry

1. Find the length of the longest side of ΔANT, with the vertices $A(-1,2)$, $N(3,5)$, $T(-1,8)$.

Solution: The absolute difference between x-coordinates of points A and N is 4, and the absolute difference between their y-coordinates is 3. Thus, $AN = \sqrt{3^2+4^2} = 5$ by the distance formula. The absolute differences between the x- and y-coordinates of N and T are 4 and 3, so $NT = 5$ as well. The absolute coordinate differences between A and T are 0 and 6, so $AT = 6$. . AT is the longest side, and the answer is $\boxed{6}$.

2. Find the distance from the point $P(3,-11)$ to the line with the equation $5x - 12y = 17$.

Solution: The distance between a point (x_1, y_1) and a line $ax+by = c$, or equivalently $ax+by-c = 0$ is $\frac{ax_1+by_1-c}{\sqrt{a^2+b^2}}$. Plugging our numbers into this formula yields that the distance is $\frac{5\cdot 3-12\cdot(-11)-17}{\sqrt{5^2+(-12)^2}} = \frac{15+132-17}{\sqrt{25+144}} = \frac{130}{\sqrt{169}} = \frac{130}{13} = 10$.

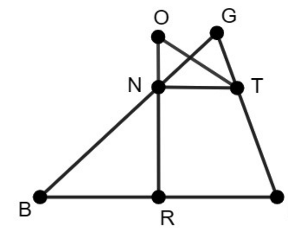
Alternatively, any line perpendicular to $5x-12y = 17$ can be expressed in the form $12x+5y = c$, and the one through point P satisfies the equation, which gives the value of c to be $12\cdot 3+5\cdot(-11) = -19$. To find the x value of the point of intersection of the lines $5x - 12y = 17$ and $12x + 5y = -19$ one can multiply the first equation by 5 and the second by 12, then add. This yields $25x - 60y = 85$ and $144x + 60y = -228$, added gives $169x = -143$, so $x = -\frac{143}{169} = -\frac{11}{13}$. Similarly solving for y, $60x - 144y = 204$ and $60x + 25y = -95$, and subtracting yields $-169y = 299$ so $y = -\frac{299}{169} = -\frac{23}{13}$. The distance between point $P(3,-11)$ and the point of intersection $(-\frac{11}{13}, -\frac{23}{13})$ is $\sqrt{(3+\frac{11}{13})^2 + (-11+\frac{23}{13})^2} = \sqrt{(\frac{39+11}{13})^2 + (\frac{-143+23}{13})^2} = \sqrt{(\frac{50}{13})^2 + (\frac{-120}{13})^2} = \frac{10\sqrt{5^2+(-12)^2}}{13} = 10$

3. Recall that a parabola is the set of points equidistant from its focus (a point) and its directrix (a line). Find the equation for the parabola with focus $(3,4)$ and directrix $y = -x$. If the equation is of the form $x^2 + Ay^2 + Bx + Cy + Dxy = E$, find $A + B + C + D + E$.

Solution: The distance from a point (x_1, y_1) on the parabola to $(3,4)$ is $\sqrt{(x_1-3)^2 + (y_1-4)^2}$ by the distance formula. The distance from the same point (x_1, y_1) and the line $y = -x$, which also can be represented as $x + y = 0$ is given by the formula above $\frac{x_1+y_1}{\sqrt{1^2+1^2}} = \frac{\sqrt{2}(x_1+y_1)}{2}$. Equating these distances and squaring both sides gives $(x-3)^2 + (y-4)^2 = \frac{(x+y)^2}{2}$. Expanding and simplifying yields $x^2 - 6x + 9 + y^2 - 8y + 16 = \frac{x^2+2xy+y^2}{2}$ and $\frac{x^2}{2} + \frac{y^2}{2} - 6x - 8y - xy = -25$. Multiplying by 2 to get it in the desired form yields $x^2 + y^2 - 12x - 16y - 2xy = -50$ so thus $A+B+C+D+E = 1 + (-12) + (-16) + (-2) + (-50) = -79$

Team Round

1. Given ΔBIG shown at right, with $\overline{BI} \parallel \overline{NT}, BN = 9, NG = 6, BI = 10, OT = 5, ON = 3$, and $BR = 3\sqrt{5}$. Find the area of quadrilateral $BITN$.



> **Solution:** First note that $\triangle BGI \sim \triangle NGT$, so $BI : NT = BG : NG = 15 : 6$ so $NT = BI \cdot NG/BG = BI \cdot 2/5 = 4$. Thus $\triangle ONT$ is a $3-4-5$ right triangle, as $ON = 3$, $NT = 4$ and $OT = 5$. This means that $\overleftrightarrow{OR} \perp \overleftrightarrow{BI}$, as $\overleftrightarrow{BI} || \overleftrightarrow{NT}$. Thus $\triangle BRN$ has a right angle at R, and the pythagorean theorem yields that $RN = \sqrt{BN^2 - BR^2} = \sqrt{81 - 45} = \sqrt{36} = 6$. Now we have both bases, $BI = 10$ and $NT = 4$, and the height, $RN = 6$, of trapezoid $BITN$. Computing the area as $h(b_1 + b_2)/2$ yields $6(4 + 10)/2 = \boxed{42}$

2. Let $f(x) = x^2$, $g(x) = 1 - x$, $h(x) = 1/x$, and $k(x) = \sin(x)$. Express $\mu(x) = 1 + \tan^2(x)$ as a compositon involving one or more of the functions f, g, h, and k.

> **Solution:** Using trig identities $\tan(x) = \sin(x)/\cos(x)$ and $\sin^2(x) + \cos^2 = 1$, the expression simplfies as follows $1 + \tan^2(x) = 1 + \frac{\sin^2(x)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \frac{1}{1-\sin^2(x)}$. This is $h(1 - \sin^2(x)) = h(g(\sin^2(x))) = h(g(f(\sin(x)))) = \boxed{h(g(f(k(x))))}$ or $\boxed{(h \circ g \circ f)(x)}$.

3. If the rational expression $\frac{2x-9}{x^2-x-6}$ is decomposed into the sum $\frac{A}{x-3} + \frac{B}{x+2}$, find the ordered pair (A, B)

> **Solution:** First, note that finding a common denominator yields $\frac{A}{x-3} + \frac{B}{x+2} = \frac{A(x+2)+B(x-3)}{x^2-x-6}$. We want to find A and B such that $A(x + 2) + B(x - 3) = 2x - 9$. If $x = -2$, then the equation simplifies to $-5B = -13$, so $B = 13/5$. Similarly, if $x = 3$ then the equation simplifies to $5A = -3$, so $A = -3/5$. Thus our ordered pair $(A, B) = \boxed{\left(\frac{-3}{5}, \frac{13}{5}\right)}$.
>
> Alternatively, we could break the equation $A(x+2)+B(x-3) = 2x-9$ into two linear equations with two variables, one equation for the x terms, and one equation for the constant terms. These would be $A+B = 2$ and $2A-3B = -9$. Adding three times the first equation to the second again yields $5A = -3$, so $A = -3/5$ and substituting into the first equation gives $B = 13/5$

4. Find the point(s) P on the x-axis such that the line through P and $(1,1)$ is perpendicular to the line through P and $(3,-8)$.

Solution: A point on the x-axis is of the form $(x,0)$. Two lines are perpendicular if their slopes are negative reciprocals. The slope through P and $(1,1)$ is $\frac{1-0}{1-x}$, and the slope of the line through P and $(3,-8)$ is $\frac{-8-0}{3-x}$. Thus we get the equation $\frac{1}{1-x} = -\frac{3-x}{-8} = \frac{3-x}{8}$. Cross multiplying yields $8 = (1-x)(3-x) = x^2 - 4x + 3$, so $x^2 - 4x - 5 = 0$ which factors as $(x-5)(x+1) = 0$. Thus $x = 5$ or $x = -1$ and so the two points are $\boxed{\text{(5,0) } \textit{and} \text{ (-1,0)}}$.

5. In how many different ways can 35¢ be made up from coins of denominations 1¢, 5¢, 10¢, and 25¢?

Solution: First, assume that a 25¢ coin is used. The remaining 10¢ could be either 1 10¢ coin, 2 5¢ coins, 1 5¢ coin and 5 1¢ coins, or 10 1¢ coins, for 4 possibilities when a 25¢ coin is used. Now if there were exactly 3 10¢ coins used, there would be 5¢ remianing, and there could either be 0 or 1 5¢ coins used, for 2 possibilities. Similarly, if there were exactly 2 10¢ coins used, there would be 15¢ remaining, and there could be from 0 to 3 5¢ coins used, for 4 for possibilities, if there was exactly 1 10¢ coins used, there would be 25¢ remaining, and there could be from 0 to 5 5¢ coins used, for 6 for possibilities, and if there were no 10¢ coins used, there would be 35¢ remaining, and there could be from 0 to 7 5¢ coins used, for 8 for possibilities. Thus there are a total of $4 + 2 + 4 + 6 + 8 = \boxed{24}$ possible combinations of those coins that add up to 35¢.

6. A stream runs due east to west. Alice is 7 miles to the north of the stream and Bill is 11 miles east and 3 miles south of Alice. What is the shortest distance Alice needs to travel in order to get to the stream and then to Bill?

Solution: Reflect Bill's position across the stream, treating the stream as a line with zero width. The distance Alice travels from her starting point to the stream and then to Bill is exactly the same as the distance she would travel along the same path to the stream and then reflected on the other side of the stream to Bill's reflected point. Clearly the shortest distance betwee Alice's starting point and Bill's reflected point is a straight line. As Bill's reflected position is 11 miles east and $7 + (7-3) = 11$ miles south of Alice's starting position, they are distance $\boxed{11\sqrt{2}}$ miles apart, which is the length of the shortest path Alice can take to Bill's that meets the stream.

7. Two men working together can build an electric car in $3\frac{3}{7}$ weeks. If one of them works alone it would take him 6 weeks. How long would it take the other man working alone?

Solution: First note that $3\frac{3}{7}$ weeks is $\frac{24}{7}$ weeks. Since the first man completes $\frac{1}{6}$ of a car per week, in that time he completed $\frac{24}{7}\frac{1}{6} = \frac{4}{7}$ of a car in that time. This means the other man completed $\frac{3}{7}$ of a car in the $\frac{24}{7}$ weeks, which is a pace of $\frac{3}{7} \div \frac{24}{7} = \frac{1}{8}$ car per week, which is the same as 1 car every $\boxed{8}$ weeks.

8. Consider the graph of an ellipse centered at $(0, a)$ and passing through the origin and $(1, a)$ for some value a. The range of values of a for which the ellipse has more than one point of intersection with the parabola $y = x^2$ can be expressed in interval notation as (c, d). Find the ordered pair (c, d).

Solution: The center of the ellipse is $(0, a)$, and its semi x-axis is length1 and its semi y-axis is length a. Thus the ellipse is given by the equation $\frac{x^2}{1^2} + \frac{(y-a)^2}{a^2} = 1$ or more simply $x^2 + \frac{(y-a)^2}{a^2} = 1$ The points of intersection with the parabola $y = x^2$ satisfy $y + \frac{(y-a)^2}{a^2} = 1$ so multiplying both sides by a^2 and expanding the square yields $a^2y + y^2 - 2ay + a^2 = a^2$ and so $y^2 + (a^2 - 2a)y = 0$ and the points of intersection are at $y = 0$ and $y = 2a - a^2$. The point $y = 2a - a^2$ only represents an actual point of intersection in addition to $y = 0$ (as opposed to an extraneous solution), when $2a - a^2 > 0$ because all points on the parabola (other than $(0, 0)$) have $y > 0$. The related equality is satisfied when $a = 0$ or $a = 2$, and the inequality is satisfied when $0 < a < 2$, or $a \in \boxed{(0, 2)}$ expressed in interval notation.

9. How many distinct ways can 7 charms be arranged on a bracelet that has a clasp on it?

Solution: Since the bracelet has a clasp on it, rotations of the charms are distinct, however, flipping the bracelet over results in an equivalent arrangement. Thus as there are 7! ways to order the charms, and this over counts by a factor of 2 because of the flipping over of the bracelet, the answer is $7!/2 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = \boxed{2520}$ distinct arrangements.

Worcester County Mathematics League
Varsity Meet 3: January 19, 2022
Answer Key

Round 1 - Sim. and the Pyth. Theorem

1. 18
2. 10
3. 1 : 6

Round 2 - Algebra I

1. 5 : 4
2. 168
3. 4

Round 3 - Functions

1. $\frac{11}{2}$
2. $\left[\frac{3}{8}, \frac{3}{2}\right]$
3. $\frac{19}{27}$

Round 4 - Combinations

1. 1024
2. $(0, 0, 6, 3)$
3. 33

Round 5 - Analytic Geometry

1. 6
2. 10
3. -79

Team Round

1. 42
2. $h(g(f(k(x))))$ *or* $(h \circ g \circ f)(x)$ (either one)
3. $\left(\frac{-3}{5}, \frac{13}{5}\right)$ (exact order)
4. $(5, 0)$ *and* $(-1, 0)$ (exact order, need both)
5. 24
6. $11\sqrt{2}$ miles
7. 8 weeks
8. $(0, 2)$ (exact order)
9. 2520